Cambridge Assessment
Admissions Testing

STEP Hints and Solutions 2017

Mathematics

STEP 9465/9470/9475

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STEP III 2017
Hints and solutions

Question 1

The first result is simply obtained by expanding the bracketed expression on the right-hand side using the definition of the binomial coefficients, and then combining the fractions using the lowest common denominator. $\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}}$ is determined by employing the first result of the question, finding that the terms telescope and then observing that ${}^{n+r}C_r \to \infty$ as $n \to \infty$, to give the answer $\frac{r+1}{r}$. The deduced result is obtained by letting $r = 2$ in the previous result and subtracting the first term of the sum in the general result. The first inequality of (ii) can be obtained by expanding ${}^{n+1}C_3$ as $\frac{n^3-n}{3!}$, observing that $\frac{n^3-n}{3!} < \frac{n^3}{3!}$ and rearranging. Similarly, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3}$ is $\frac{-480}{n^3(n^2-1)(n^2-4)}$ which is negative as $n \geq 3$ and so the denominator is positive, leading to the second inequality. The first numerical inequality in the final result is obtained from the second inequality of part (ii) using the final result of (i) for the first summed term, the penultimate result of (i) for the second summed term (adjusting the index over which it is summed), and including the terms for $n = 1$ and $n = 2$. The second numerical inequality in the final line is obtained from the first inequality of part (ii), again including the two extra terms and using the final result of (i).

Question 2

(i) is simply obtained by applying $e^{i\theta}$ to $z - a$. Using the result of (i) twice for SR and for a rotation about c and equating, both sides can be multiplied by $-e^{\frac{-i(\varphi+\theta)}{2}}$. In the case $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found, and then SR is a translation by $(b-a)\left(1 - e^{i\varphi}\right)$. If $RS = SR$, and if $\varphi + \theta = 2n\pi$, then

$(b-a)\left(1 - e^{i\varphi}\right) = (a-b)\left(1 - e^{i\theta}\right)$ and so either $a = b$ or if $a \neq b$, $\theta = 2m\pi$. If $\varphi + \theta \neq 2n\pi$, $a = b$, $\theta = 2n\pi$, or $\varphi = 2n\pi$

Question 3

Writing down the sum of the three roots of the cubic gives an expression which must be q in the quartic and $-A$ in the cubic. In the specific case, the cubic equation is

$y^3 - 3y^2 - 40y + 84 = 0$ which has roots -6, 2 and 7, and thus $\alpha\beta + \gamma\delta = 7$. Expanding the expression whose value is required in (ii) gives q without the two terms whose sum has just been found, and hence -4. Equally the product of those two terms is s (10), and so a quadratic equation for them gives $\alpha\beta = 5$, the larger of the two roots. Using the first result of (ii) with the knowledge that $(\alpha + \beta) + (\gamma + \delta) = p = 0$, and that

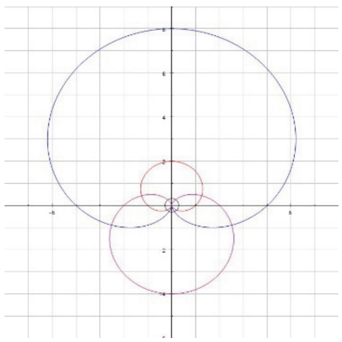
$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = 6$, leads to the results that α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of $t^2 - 2t + 2 = 0$. Hence the four roots of the quartic are $1 \pm i$, $-1 \pm 2i$.

Question 4

Letting $\log_a \mathrm{f}(x) = z$, $\mathrm{f}(x) = a^z = \mathrm{e}^{z \ln a}$ (using the result from the formula book) and so, $\ln \mathrm{f}(x) = z \ln a = \ln a \ \log_a \mathrm{f}(x)$, which substituted in the geometric mean definition simplifies rapidly to the required result. Part (ii) can be obtained by substituting for $\mathrm{h}(x)$ in the expression for $\mathrm{H}(y)$ and then manipulating using the logarithm of a product. Part (iii) can be obtained using the result of part (i) with $a = b$, and then checking separately that the simple case $b = 1$ works. In part (iv), setting the defined expression for the geometric mean of $\mathrm{f}(x)$ equal to $\sqrt{\mathrm{f}(y)}$, and taking the logarithm of both expressions yields, after minor rearrangement, $\int_0^y \ln \mathrm{f}(x) \mathrm{d}x = \frac{y}{2} \ln \mathrm{f}(y)$. Differentiating this with respect to y, and again rearranging, leads to the first order separable differential equation $\frac{\mathrm{f}'(y)}{\mathrm{f}(y) \ln \mathrm{f}(y)} = \frac{1}{y}$, which integrates to $\ln \ln \mathrm{f}(y) = \ln y + c$ leading to the desired result by judicious choice of c.

Question 5

Converting polar coordinates to Cartesian and differentiating each of x and y with respect to θ gives a fraction which simplifies to $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{f}+\mathrm{f}'\tan\theta}{-\mathrm{f}\tan\theta+\mathrm{f}'}$ once the numerator and denominator have been divided by $\cos\theta$. Setting the product of two such expressions for f and g equal to negative 1 and simplifying leads to $(\mathrm{fg}+\mathrm{f}'\mathrm{g}')\sec^2\theta = 0$ and hence the desired result. Substituting for g in the displayed result and solving the resulting first order differential equation for f (by integrating factor or separation of variables) leads to $\mathrm{f}(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ which simplifies by eliminating $\cos^2\theta$ in favour of $\sin^2\theta$, and hence using the given point, $\mathrm{f}(\theta) = 2(1-\sin\theta)$.



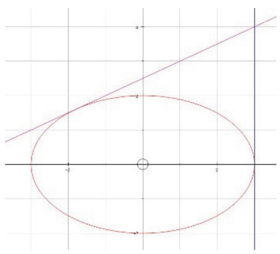
Question 6

The appropriate substitution for part (i) was $u = v^{-1}$, and having changed variable the resulting integral has limits x^{-1} and ∞, which can be expressed as the difference of two integrals with these as their upper limits and zero as their lower. To obtain $\frac{\mathrm{d}v}{\mathrm{d}u}$ in part (ii), one method is to make a (the constant) the subject of the formula and to differentiate with respect to u; an alternative is to differentiate v directly, to multiply numerator and denominator by $(1+u^2)$, to expand the numerator and then to express it as $(1-au)^2 + (u+a)^2$ leading to the desired result. Applying this substitution to the defined $\mathrm{T}(x)$, results in an integral which can be expressed again as the difference of two integrals as in part (i). Taking the result of (i) and rearranging to make $\mathrm{T}(x^{-1})$ the subject, $\mathrm{T}(x)$ can be substituted for using the result just found and in turn $\mathrm{T}(a)$ can be replaced using the result of (i) with x as a. The final result of (ii) is achieved by letting $y = x^{-1}$, and $b = a^{-1}$ in that just found. Throughout, it is important that the conditions expressed as inequalities are substantiated. To find $\mathrm{T}(\sqrt{3})$ in (iii), apply the final result of (ii) letting $y = b = \sqrt{3}$, whereas to find $\mathrm{T}(\sqrt{2}-1)$, let $x = \sqrt{2}-1$ and $a = 1$ in the result

$\mathrm{T}(x) = \mathrm{T}\left(\frac{x+a}{1-ax}\right) - \mathrm{T}(a)$, deal with $\mathrm{T}(\sqrt{2}+1)$ by letting $x = \sqrt{2}+1$ in the result of part (i) and then $\mathrm{T}(1)$ using the same result but with $x = 1$.

Question 7

Showing T lies on the ellipse is merely a matter of substituting T's coordinates into the left-hand side of the ellipse equation and simplifying to equal 1. The first result of (i) is tantamount to finding the equation of the tangent at T. Parametric differentiation leads to $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{b(1-t^2)}{2at}$ giving L as $y - \frac{2bt}{(1+t^2)} = -\frac{b(1-t^2)}{2at}\left(x - \frac{a(1-t^2)}{(1+t^2)}\right)$ which (X, Y) satisfies, thus simplifying to the required result. The deduction can be made by requiring that the discriminant of the quadratic in t is positive and geometrically (X, Y) is a point outside the ellipse. Relaxing the restriction on the value of X, $X = \pm a$, so the inequality implies $Y \neq 0$ and thus there is a vertical tangent and another with one possible configuration as shown.



The first result of (ii) is obtained by considering p and q to be the roots of the quadratic in t, and hence being able to write down their product. Similarly, $p + q = \frac{2aY}{(a+X)b}$. The final result is obtained by finding expressions for y_1 and y_2 in terms of p and q respectively (without loss of generality), imposing the condition on y_1 and y_2 to get an equation in p and q, and then using the first two results from this part of the question to substitute for the product and sum of p and q.

Question 8

The stem can be achieved by adding the two summations, expanding the brackets, and observing that the resulting two summed terms telescope. (i) is simply a case of using the given expression for b_m , and letting $a_m = 1$ (or any constant) in the stem result, simplifying the left-hand side using the given note, and dividing through by $\sin\frac{1}{2}x$. Part (ii) can be obtained by letting $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$ (or similarly $b_m = \cos\left(m - \frac{1}{2}\right)x$) in the stem which simplifies to give $p = -\frac{1}{4}n$ and $q = \frac{1}{4}(n+1)$.

Question 9

The first result can be obtained by writing the equation of motion for each particle separately and then adding the two equations to eliminate the unknown tension; two integrations with respect to time complete the working once the constants of integration (both zero) are evaluated. Using the result obtained, at the time T (given) with the value of x as a, y works out to also be a. As a consequence, conservation of energy has no elastic energy term at that instant, merely kinetic energy for each particle and the lost potential energy of A. Combining this equation with that obtained after the first integration in the initial result of the question gives simultaneous equations for the two speeds at that instant, and substituting for the speed of A gives a quadratic with the desired result as its repeated root.

Question 10

The first result is obtained by conserving energy for the rod and particle together (rotational kinetic energy and gravitational potential energy) and simplifying the algebra. Differentiating that result with respect to time and then simplifying gives $2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$. Alternatively, the same result can be obtained by taking moments about an axis through P. Resolving perpendicular to the rod for the particle and rearranging the equation generated yields an expression for the normal reaction, $mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$, having used the previously obtained expression for $\ddot{\theta}$. This is demonstrably positive under the given conditions. Resolving along the rod towards P (i.e. radially inwards) yields an expression for the friction which is simplified using the first obtained result of the question, and then applying the conditions for limiting friction yields the given result. In the case $l > 2a$, the particle loses contact immediately as the rod falls away quicker than the particle accelerates downwards; this can be shown either by considering the equation of rotational motion for the rod alone about P and finding $l\ddot{\theta} = \frac{lg}{2a}$, or by observing from previous working that the normal reaction of the rod on the particle would need to be negative for the particle to stay in contact with the rod.

Question 11

Conserving linear momentum in part (i) leads to $u = \frac{nmv}{M}$, and using this result leads directly to the displayed kinetic energy result. Conserving momentum before and after the r^{th} gun is fired gives $\big(M + \big(n - (r-1)\big)m\big)u_{r-1} = (M + (n-r)m)u_r - m(v - u_{r-1})$ which leads to the required result, and summing that result for $r = 1$ to n gives, on the right-hand side, a sum of n terms, each of which can be shown to be less than (or equal to in one case) $\frac{mv}{M}$, and hence the result. For (iii), considering the energy of the truck and the $\big(n - (r-1)\big)$ projectiles before and after the r^{th} projectile is fired,

$$K_r - K_{r-1} = \tfrac{1}{2}(M + (n-r)m){u_r}^2 + \tfrac{1}{2}m(v - u_{r-1})^2 - \tfrac{1}{2}\big(M + \big(n - (r-1)\big)m\big){u_{r-1}}^2 .$$

Simplifying this by collecting the terms $\frac{1}{2}(M + (n-r)m)({u_r}^2 - {u_{r-1}}^2)$ leads to the printed result via the difference of two squares factorisation and use of the result from (ii). Once again, summing as before with telescoping terms leads to the second printed result, and the final inequality follows using the inequality from (ii) via a little algebraic simplification; the final step is quite a slack inequality.

Question 12

To obtain the first result of (i), sum in turn over x and y from 1 to n to obtain the total probability (1!) yields $k = \frac{1}{n^2(n+1)}$, and then sum over y. For independence, it would be necessary that $\mathrm{P}(X = x, Y = y) = \mathrm{P}(X = x)\mathrm{P}(Y = y)$ and it is possible to simplify the algebra of this equation having substituted for each probability to see that there are numerous examples for which this does not hold. Proceeding as at the start of the question to obtain $\mathrm{E}(XY) = \frac{(n+1)(2n+1)}{6}$ and summing over x using the printed result of (i) to find $\mathrm{E}(X) = \frac{(7n+5)}{12}$ (and hence $\mathrm{E}(Y)$ also) yields, following simplification, $Cov(X, Y) = \frac{-(n-1)^2}{144}$, establishing the required result.

Question 13

To find $\mathrm{V}(x) = \sigma^2 + (x - \mu)^2$, expanding the definition of $\mathrm{V}(x)$ and then expressing $\mathrm{E}(X^2)$ in terms of variance yields the result. The result for $\mathrm{E}(Y)$ that is given in the question follows directly from the first result, and that $\mathrm{V}(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2$ in the uniform case, follows from previous working, applying standard knowledge of the mean and variance of the uniform distribution. In order to find the probability density function of Y, it is simplest to find the cumulative distribution function of Y first which is done by algebraic rearrangement of $\mathrm{P}\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 < y\right)$ and then differentiation to give $\mathrm{f}(y) = \left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$, $\frac{1}{12} \leq y \leq \frac{1}{3}$ and 0 otherwise. The final verification is conducted by integration of $y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$ with suitable limits, which can be done by numerous methods such as expressing the function as $\left(y - \frac{1}{12}\right)\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$ or change of variable, of which, letting $u^2 = y - \frac{1}{12}$ is just one example.